University of Groningen
Exam Numerieke Wiskunde 1, July 11, 2014

Use of a simple calculator is allowed. All answers need to be motivated.
In front of the exercises you find its weight. In fact it gives the number of tenths which can be gained in the final mark. In total 6.0 points can be scored with this exam.

4

1. (a) $\boxed{4}$ Compute the LU factorization of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

with L and U a 2×2 nonsingular lower and upper triangular matrices respectively.

(b) Consider solving the linear system $Ax = b$ using the Gauss-Seidel method

2

i. $\boxed{2}$ Derive this iterative method.

13 3

ii. $\boxed{3}$ Also derive the recurrence relation for the error and indicate the iteration matrix.

15 2

iii. $\boxed{2}$ Give the necessary and sufficient requirement that must be satisfied for convergence.

1

(c) i. $\boxed{4}$ A matrix has eigenvalues $-2, -\frac{1}{2}, 0, \frac{2}{3}$. To which eigenvalue-eigenvector pair the power method will converge? Show this by an analysis.

2

ii. $\boxed{2}$ What is the rate of convergence of the power iteration in the the previous part?

18

2. Consider the fixed point method $x_{n+1} = g(x_n)$ with fixed point p.

4

(a) $\boxed{4}$ Using Taylor series show that for x_n close to p it holds that

$$x_{n+1} - p = g'(p)(x_n - p) + \frac{1}{2} g''(p)(x_n - p)^2 + \ldots \qquad (1)$$

1

(b) $\boxed{2}$ When will we have linear convergence and when quadratic?

1

(c) $\boxed{4}$ Give an estimate of the error $x_{n+1} - p$ expressed in x_{n-1}, x_n, and x_{n+1} which are very close to p in case $g'(p) \neq 0$.

(d) $\boxed{3}$ Suppose that x and $f(x)$ are vectors of length 2. Derive the variant of (1) for this case. You do not need to give the quadratic term explicitly.

24

Continue on other side!

3. (a) $\boxed{4}$ Derive the quadratic interpolation polynomial for $f(x) = x^3 - 5x^2 + 100x + 5$ using Lagrange basis functions on three interpolation points 0, 1 and 2.

   (b) $\boxed{3}$ Give the general form of the interpolation error when 3 interpolation points are used. Determine with this an upperbound for the interpolation error on [0,2] (given is that $|x(x-1)(x-2)| \leq \frac{2}{9}\sqrt{3}$ on [0,2]).

   (c) i. $\boxed{2}$ Give the midpoint rule for integration on the interval $[a, b]$.

   ii. $\boxed{2}$ What is the degree of exactness of the midpoint rule and show this.

   (d) $\boxed{3}$ Show that the results in the table below converge second order in h.

h	I(h)
1	1.42521
0.5	1.41678
0.25	1.414842
0.125	1.414370

4. Consider on [0, 1] for $u(x,t)$ the diffusion equation $\partial u/\partial t = \partial^2 u/\partial x^2 + x\exp(-t)$ with initial condition $u(x,0) = \sin(\pi x)$ and boundary conditions $u(0,t) = \sin^2(t)$ and $u(1,t) = 0$. Let the grid in x-direction be given by $x_i = i\Delta x$ where $\Delta x = 1/m$.

   (a) $\boxed{3}$ Show that $\frac{\partial^2 u}{\partial x^2}(x_i, t) = \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{\Delta x^2} + O(\Delta x^2)$.

   (b) $\boxed{5}$ Show that the system of ordinary differential equations (ODEs) that results from using the expression in (a) is of the form

$$\frac{d}{dt}\mathbf{u}(t) = -\frac{2}{\Delta x^2}(I - B)\mathbf{u}(t) + \mathbf{b}(t)$$

   and give B, $\mathbf{b}(t)$ and $\mathbf{u}(0)$.

   (c) $\boxed{2}$ Derive and sketch in the complex plane the location of the eigenvalues of B.

   (d) $\boxed{2}$ Derive from the previous part that the eigenvalues of $-\frac{2}{\Delta x^2}(I - B)$ are located in the interval $[-\frac{4}{\Delta x^2}, 0]$.

   (e) $\boxed{3}$ Derive the region of absolute stability of the trapezoidal method.

   (f) $\boxed{1}$ Use the results of the last two parts to show that there is no time step restriction if we apply the trapezoidal method to the system of ODEs in part (b).

Total $\boxed{60}$